Comment on "Helical MRI in magnetized Taylor-Couette flow"

G. Rüdiger
Astrophysikalisches Institut Potsdam, An der Sternwarte 16, D-14482 Potsdam, Germany

R. Hollerbach
Department of Applied Mathematics, University of Leeds, Leeds, LS2 9JT, United Kingdom

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Liu *et al.* [Phys. Rev. E **74**, 056302 (2006)] have presented a WKB analysis of the helical magnetorotational instability (HMRI), and claim that it does not exist for Keplerian rotation profiles. We show that if radial boundary conditions are included, the HMRI can exist even for rotation profiles as flat as Keplerian, provided only that at least one of the boundaries is sufficiently conducting.

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The helical magnetorotational instability (HMRI) [1, 2, 3, 4, 5], is similar to the standard magnetorotational instability (SMRI) [6, 7, 8], in the sense that both are mechanisms whereby hydrodynamically stable differential rotation profiles may be destabilized by the addition of magnetic fields. However, the way in which they behave in the limit of small magnetic Prandtl number Pm is very different. Unlike the SMRI, which ceases to exist for zero Pm, the HMRI continues to exist, with the relevant nondimensional parameters being the hydrodynamic Reynolds number Re and the Hartmann number Ha, but the magnetic Reynolds number $\mathrm{Rm} = \mathrm{Pm} \cdot \mathrm{Re}$ and the Lundquist number $\mathrm{S} = \mathrm{Pm}^{1/2} \cdot \mathrm{Ha}$ (the relevant parameters for the SMRI) tending to zero along with Pm.

Given these very different scalings, one natural question to ask is whether they exist for the same range of rotation profiles. The SMRI is known to operate for any outwardly decreasing profile. In contrast, the HMRI is more delicate; the results of [1] indicate that as one moves beyond the Rayleigh line to ever flatter profiles, one eventually switches back to the SMRI scalings. Liu *et al.* [9] explored this issue more systematically, and claim that the HMRI cannot exist for profiles as flat as Keplerian. Specifically, their WKB analysis indicates that if $\Omega \sim r^n$, then the HMRI requires $n < -1.66$, thereby excluding the Keplerian value $n = -1.5$. [Their Eq. (12), with their $Ro = n/2$.]

We do not dispute the validity of their analysis; indeed, some of our results below are in excellent agreement with it. However, we note that any WKB analysis is necessarily local, and does not incorporate the boundary conditions of the global problem. We show here that if the radial boundary conditions are taken into account, the HMRI can exist even for the Keplerian value $n = -1.5$, provided only that one of the boundaries is at least somewhat conducting.

The eigenvalue problem to be solved is

$$\mathrm{Re}\,\gamma\, v = D^2 v + \mathrm{Re}\, ik\, r^{-1}(r^2\Omega)'\,\psi + \mathrm{Ha}^2\, ik\, b,$$

$$\mathrm{Re}\,\gamma\, D^2\psi = D^4\psi - \mathrm{Re}\, 2ik\,\Omega\, v + \mathrm{Ha}^2(k^2\psi + 2ik\beta r^{-2} b),$$

$$0 = D^2 b + ik\, v - 2ik\,\beta r^{-2}\psi,$$

essentially the same as in [1], except that we restrict attention to the $\mathrm{Pm} \to 0$ limit, ensuring that any instabilities we obtain will necessarily be the HMRI. (Priede *et al.* [10] have very recently also considered this $\mathrm{Pm} \to 0$ limit; some of their results are quite relevant here, and will be discussed below.)

The boundary conditions associated with v and ψ are no-slip, just as in [1]. The boundary conditions associated with b are

$$b = \epsilon (rb)' \quad \text{at} \quad r_{\mathrm{i}} = 1, \qquad b = 0 \quad \text{at} \quad r_{\mathrm{o}} = 2.$$

The outer boundary is therefore insulating, whereas the nature of the inner boundary depends on ϵ: $\epsilon = 0$ is insulating, $\epsilon = \infty$ is perfectly conducting. Intermediate values correspond to a boundary consisting of a thin layer of relative conductance ϵ (see for example [11, 12] for this thin boundary layer approximation in other contexts).

$\Omega(r)$ is the rotation profile whose stability is to be investigated. In [1, 2] we considered Taylor-Couette profiles of the form $\Omega = c_1 + c_2/r^2$. To facilitate comparison with Liu *et al.*, here we will primarily consider profiles of the form $\Omega = r^n$. As we will see though, the two choices yield almost identical behavior.

Figure 1 shows contour plots of the critical Reynolds number for the onset of the HMRI, as a function of n and β, and optimized over the Hartmann number Ha and the axial wavenumber k. Turning first to the $\epsilon = 0$ plot on the left, we note how increasing β from 1 to 5 facilitates the instability, that is, allows it to exist increasingly far beyond the Rayleigh line at $n = -2$. Beyond $\beta \approx 5$ though the Re_{c} curves become largely independent of β. And crucially, even the $\mathrm{Re}_{\mathrm{c}} = 10^6$ curve asymptotes just to the left of the Liu *et al.* line at $n = -1.66$. These $\epsilon = 0$ results are therefore in excellent agreement with their prediction that the HMRI only exists to the left of this line.

However, if we now turn to the $\epsilon = \infty$ plot on the right, we note that $\mathrm{Re}_{\mathrm{c}} = 10^4$ already extends beyond the Liu *et al.* line, and $Re_c = 10^5$ extends beyond the Kepler line $n = -1.5$. Simply switching the inner boundary from insulating to conducting is sufficient to allow the HMRI to operate even for Keplerian rotation profiles. Similar

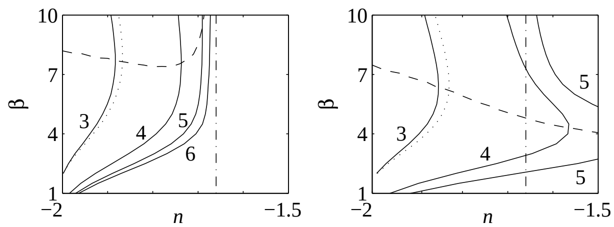

FIG. 1: The left panel is for $\epsilon = 0$, the right for $\epsilon = \infty$. In each case the solid curves are contours of $\log \mathrm{Re_c}$ as a function of n and β, optimized over Ha and k. The dashed curves are the location where $\mathrm{Re_c}$ is optimized over β as well. The dotted curves show how the 10^3 contours are shifted slightly to the right if the r^n profile is replaced by a Taylor-Couette profile having the same Ω_o/Ω_i ratio. The dash-dotted lines denote the $n = -1.66$ boundary, beyond which the HMRI does not exist in the Liu *et al.* analysis. Note how this agrees very well with our results for $\epsilon = 0$, but not at all for $\epsilon = \infty$.

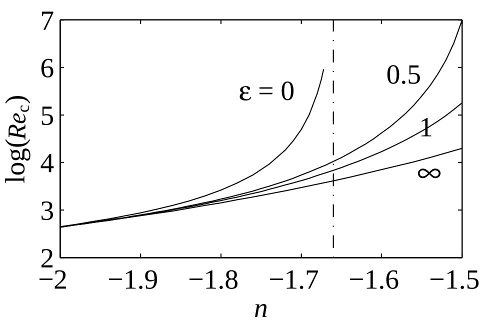

FIG. 2: $\log \mathrm{Re_c}$ as a function of n, optimized over k, Ha and β. Note how $\epsilon = 0$ does indeed appear to reach a vertical asymptote at the Liu *et al.* value $n = -1.66$, but $\epsilon = 0.5$ is already sufficiently large to reach the Kepler value $n = -1.5$.

results are obtained if instead it is the outer boundary that is switched from insulating to conducting. Why the electromagnetic boundary conditions should have such a dramatic effect is not clear, but it is certainly well known in many other contexts, e.g. [13, 14], that they can play a crucial role.

Note also that exactly the same phenomenon illustrated in Fig. 1 is already implicit in Fig. 2 of [10], where the HMRI exists up to $\mu \approx 0.32$ for insulating boundaries, but up to $\mu \approx 0.45$ for conducting boundaries, where $\mu = \Omega_o/\Omega_i$. Translating from μ to n via $\mu = 2^n$, their results become $n \approx -1.64$ for insulating boundaries, versus $n \approx -1.15$ for conducting boundaries. That is, the Keplerian value $n = -1.5$ is accessible with conducting boundaries, but not with insulating ones.

At first sight it would appear that their value of $n \approx -1.64$ for insulating boundaries is already in conflict with the Liu *et al.* limit $n < -1.66$. In fact, this slight discrepancy is due to the difference between the $\Omega = c_1 + c_2/r^2$ profile used by [10], and the $\Omega = r^n$ profile considered by Liu *et al.* Even if c_1 and c_2 are chosen to match $\Omega_i = 1$ and $\Omega_o = 2^n$, a Taylor-Couette profile will be somewhat steeper near r_i, and correspondingly somewhat shallower near r_o. By concentrating near the inner boundary, the instability can then operate at somewhat larger Ω_o/Ω_i values than for an r^n profile. The dotted lines in Fig. 1 quantify this effect, showing the $\mathrm{Re_c} = 10^3$ curves if the r^n profile is replaced by a Taylor-Couette profile, with c_1 and c_2 chosen as indicated above. We see that qualitatively the two profiles yield exactly the same behavior, but that the Taylor-Couette profile extends to slightly larger values of n. It was precisely to avoid this effect, and thereby allow a direct comparison with Liu *et al.*, that we chose here to concentrate on the r^n profiles.

The dashed lines in Fig. 1 show the locations along which $\mathrm{Re_c}$ is optimized not only over k and Ha, but over β as well. Fig 2 shows $\mathrm{Re_c}$ as a function of n along these lines, now including not just $\epsilon = 0$ and ∞, but also $\epsilon = 0.5$ and 1. We see that $\epsilon = 0.5$ is already large enough to reach the Kepler value $n = -1.5$. That is, if the conductance of the inner boundary is only 1/2 that of the fluid region, the HMRI already exists for rotation profiles as flat as Keplerian.

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